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A NOTE ON HAUSDORFF SEPARATION

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The examples usually given as instances of topological spaces that have T_1-separation but not T_2-separation (Hausdorff) also have the property that some compact subset is not closed. This with the classic result concerning closedness of compact subsets of a Hausdorff space suggests the question of the equivalence of Hausdorff separation and the condition that the class of compact subsets be a subclass of the class of the closed subsets of a given space. The following is a simple result of this type and may be of some use in an introductory course in point set topology.

THEOREM. *If X is a space satisfying the first axiom of countability, then a necessary and sufficient condition that X be a Hausdorff space is that the class of compact subsets of X be a subclass of the class of closed subsets of X.*

Only the sufficiency need be considered here.

Since points are compact, it is immediate that X must be at least T_1. Also it can be assumed that the neighborhood base $\{V_n/n=1, 2, \cdots\}$ at each point is such that $V_n \subseteq V_m$, $n \geqq m$. Suppose there exist points x and y such that there are no disjoint pairs of neighborhoods of x and y respectively. Then a sequence $\{x_n/n=1, 2, \cdots\}$ may be selected by choosing each x_n in the intersection of the nth sets of the neighborhood bases of x and y. The set $\{x_n/n=1, 2, \cdots\} \cup [x]$ is compact but is not closed since y is an accumulation point.

That the assumption of a local countable base or some other restriction is necessary is seen from the following example.

Let X be an uncountable set with a topology such that a set is open if and only if it is X, the null set or the complement of a countable set. The space is not Hausdorff and does not satisfy the first axiom of countability. However the only compact sets are finite sets and hence closed.

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